2SLS

1st Stage: Estimate the values of the end. exp. var.s by regressing them to all the P.D. var.s of the model.

2nd Stage: End. exp. var.s are replaced by the estimate obtained in the 1st step. Then, the coefficients are estimated by regressing the dep. var. to the estimated values of the end. exp. var.s & the P.D. exp. var.s.

It is expected that when we substitute end. exp. var.s by their estimates, they become asymptotically independent of the disturbances.

Let us take one equ^n of an SEM:

$$y = Y_1\beta + X_1\gamma + u$$

$$\text{or} \quad y = \begin{bmatrix} Y_1 & X_1 \end{bmatrix} \begin{bmatrix} \beta \\ \gamma \end{bmatrix} + u$$

$$\text{or} \quad y = Z\delta + u \longrightarrow (1)$$

Stage I: Estimate Y_1 by regressing it to all P.D. var.s (X).

$$\hat{Y}_1 = X\hat{\pi}_1 = X(X'X)^{-1}X'Y_1 \longrightarrow (2)$$

$$\left[\begin{aligned} &By_t + \Gamma x_t = u_t \\ &y_t = -B^{-1}\Gamma x_t + B^{-1}u_t \\ &\quad = \pi x_t + v_t \\ &\hat{y}_t = \hat{\pi} x_t \end{aligned}\right.$$

Stage II:

$$y = \hat{Y}_1\beta + X_1\gamma + u^*$$

$$= \begin{bmatrix} \hat{Y}_1 & X_1 \end{bmatrix} \begin{bmatrix} \beta \\ \gamma \end{bmatrix} + u^*$$

[u^* because some errors are omitted by estimating Y_1]

$$\left[\begin{aligned} &y = X\beta + u \\ &\hat{y} = X\hat{\beta} = X(X'X)^{-1}X'y \\ &\hat{Y}_1 = X(X'X)^{-1}X'Y_1 \end{aligned}\right.$$

$$\text{or,} \quad y = \hat{Z}\delta + u^*$$

$$\therefore \hat{\delta}_{2SLS} = (\hat{Z}'\hat{Z})^{-1}\hat{Z}'y$$

Expanding we get,

$$\begin{bmatrix} \hat{\beta}_{2SLS} \\ \hat{\gamma}_{2SLS} \end{bmatrix} = \begin{bmatrix} \hat{Y}_1'\hat{Y}_1 & \hat{Y}_1'X_1 \\ X_1'\hat{Y}_1 & X_1'X_1 \end{bmatrix}^{-1} \begin{bmatrix} \hat{Y}_1'y \\ X_1'y \end{bmatrix} \longrightarrow (3)$$

$$\hat{Z}'\hat{Z} = \begin{bmatrix} \hat{Y}_1' \\ X_1' \end{bmatrix} \begin{bmatrix} \hat{Y}_1 & X_1 \end{bmatrix} = \begin{bmatrix} \hat{Y}_1'\hat{Y}_1 & \hat{Y}_1'X_1 \\ X_1'\hat{Y}_1 & X_1'X_1 \end{bmatrix} \qquad \hat{Z}'y = \begin{bmatrix} \hat{Y}_1' \\ X_1' \end{bmatrix} y = \begin{bmatrix} \hat{Y}_1'y \\ X_1'y \end{bmatrix}$$

$\left[Y_1 = \hat{Y}_1 + \hat{V}_1 \longrightarrow \text{mat}^x \text{ of OLS residuals from the 1st stage estimation of } Y_1 \right]$ ②

From the 1st stage, the mat^x of residuals will be

$$\hat{V}_1 = Y_1 - \hat{Y}_1$$

$$\Rightarrow \hat{Y}_1 = Y_1 - \hat{V}_1$$

Now, $\hat{Y}_1'\hat{Y}_1 = \hat{Y}_1'[Y_1 - \hat{V}_1]$

$$= \hat{Y}_1' Y_1 \longrightarrow (i)$$

$[\hat{Y}_1'\hat{V}_1 = \text{null mat}^x$

[Because of properties of OLS residuals, elements will be zero].

$y = x\beta + u \quad \rightarrow u'u = y'y - y'x\beta - \beta'x'y + \beta'x'x\beta$

1st differentiation

$-2x'y + 2x'x\beta = 0$

$\Rightarrow \cancel{x'x\beta - x'y} \Rightarrow -x'(y - x\hat{\beta}) = 0$

$\Rightarrow x'\hat{u} = 0$

$\hat{Y}_1 = x(x'x)^{-1}x'Y_1$

$\therefore \hat{Y}_1' = Y_1'x(x'x)^{-1}x'$

$\hat{Y}_1'\hat{V}_1 = Y_1'x(x'x)^{-1}\underbrace{x'\hat{V}_1}_{0} = \text{Null}$

$$x_1'\hat{Y}_1 = x_1'(Y_1 - \hat{V}_1)$$

$$= x_1'Y_1 - \underbrace{x_1'\hat{V}_1}_{0}$$

$$= x_1'Y_1 \longrightarrow (ii)$$

& $\hat{Y}_1' x_1 = Y_1' x_1 \longrightarrow (iii)$

Substituting (i), (ii) & (iii) in (3),

$$\begin{bmatrix} \hat{\beta}_{2SLS} \\ \hat{\gamma}_{2SLS} \end{bmatrix} = \begin{bmatrix} \hat{Y}_1'Y_1 & Y_1'x_1 \\ x_1'Y_1 & x_1'x_1 \end{bmatrix}^{-1} \begin{bmatrix} \hat{Y}_1'y \\ x_1'y \end{bmatrix} \longrightarrow (4)$$

$$= \begin{bmatrix} Y_1'x(x'x)^{-1}x'Y_1 & Y_1'x_1 \\ x_1'Y_1 & x_1'x_1 \end{bmatrix}^{-1} \begin{bmatrix} Y_1'x(x'x)^{-1}x'y \\ x_1'y \end{bmatrix} \longrightarrow (5)$$

[Principle is completed in (3). But we do the manipulations for convenience.]

By the proof of asymptotic independent relation, we know that, 2SLS estimators are equivalent to IV estimators if the 1st stage estimate of end. exp. var.s are used as instrument for the end. exp. var.s.

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If 'W' is the mat^x of IVs,

$$W = [\hat{Y}_1 \quad X_1]$$

$$\hat{\delta}_{IV} = (W'Z)^{-1} W'y$$

or, $$\begin{bmatrix} \hat{\beta}_{IV} \\ \hat{\gamma}_{IV} \end{bmatrix} = \begin{bmatrix} \hat{Y}_1'Y_1 & \hat{Y}_1'X_1 \\ X_1'Y_1 & X_1'X_1 \end{bmatrix}^{-1} \begin{bmatrix} \hat{Y}_1'y \\ X_1'y \end{bmatrix}$$

$$= \begin{bmatrix} \hat{\beta}_{2SLS} \\ \hat{\gamma}_{2SLS} \end{bmatrix} \quad \because \hat{Y}_1'X_1 = Y_1'X_1 \text{ in (iii)}$$

$y = Z\delta + u$

N. Eqn $- Z'y + Z'Z\delta = 0$

$-Z'(y - Z\delta) = 0$

$\underset{W}{=}$ $\Rightarrow W'(y - Z\delta) = 0$

Thus, 2SLS E will be equivalent to IV E if . . . . . .

also the 2SLS E will be consistent if,

$$\text{plim } \tfrac{1}{n} W'u = 0$$

or. $$\text{plim } \frac{1}{n} \begin{bmatrix} \hat{Y}_1'u \\ X_1'u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$W'u = \begin{bmatrix} \hat{Y}_1' \\ X_1' \end{bmatrix} u = \begin{bmatrix} \hat{Y}_1'u \\ X_1'u \end{bmatrix}$$

or, $$\text{plim} \begin{bmatrix} \frac{1}{n} \hat{Y}_1'u \\ \frac{1}{n} X_1'u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now, $$\text{plim } \tfrac{1}{n} \hat{Y}_1'u = \text{plim } \tfrac{1}{n}[Y'X(X'X)^{-1}X'u]$$

$$= \text{plim}\left[\tfrac{1}{n} Y'X\,(X'X)^{-1}.\ \tfrac{1}{n} X'u\right]$$

$$= \text{plim } \tfrac{1}{n} Y'X .\ \text{plim}\left(\tfrac{1}{n} X'X\right)^{-1}.\ \text{plim } \tfrac{1}{n} X'u$$

$$= 0 \quad \text{if plim } \tfrac{1}{n} X'u = 0$$

This is automatically satisfied if X contains only exo. var.s as exo. var.s are asymptotically independent of disturbances. But if X contains lagged end. var.s also, the cond^n will be satisfied only if the disturbances themselves are serially independent.

Note: (1) ILS can be applied when the equ^{ns} of an SEM is exactly identified. But cannot be applied in case of overidentification. However, 2SLS can be used in case of over-identification also.

(2) For exactly identified equ^{ns}, ILS & 2SLS are equivalent.